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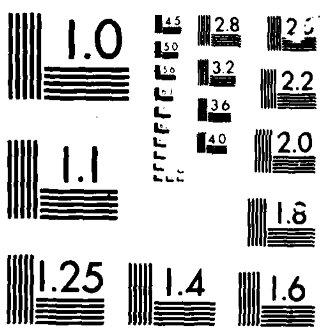

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SOLITARY AND PERIODIC WAVES IN SWIRLING FLOW

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ABSTRACT

Solitary and periodic internal waves are shown to exist in swirling flow. Incompressible, inviscid fluids in a right cylinder of infinite length and finite radius are considered. Variational techniques are used to demonstrate that the Euler equations possess solutions that represent progressing waves of permanent form. Moreover, internal solitary wave solutions are shown to arise as the limiting forms of internal periodic waves as the period length becomes unbounded.

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SIGNIFICANCE AND EXPLANATION

The study of vortex breakdown gives rise to an interest in waves in swirling flow. Our interest is centered on the existence of both solitary and periodic internal waves. In this report a model physical problem is studied in a mathematically exact formulation. We restrict our attention to an incompressible, inviscid fluid swirling through a right cylinder of infinite length and finite radius. Our theory, which is not restricted to small amplitudes, predicts both waves of elevation and depression, depending on the angular velocity (swirl) distribution and the velocity distribution at infinity. Just as for the classical surface solitary waves, these internal solitary waves are single-crested, symmetric, and decay exponentially away from the crest. Hence they represent disturbances of essentially finite extent. Variational techniques and the theory of rearrangements are used to demonstrate these qualitative features. Moreover, we show that the solitary internal wave arises as a limit of periodic internal waves of increasing wave lengths.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SOLITARY AND PERIODIC WAVES IN SWIRLING FLOW

Scott A. Markel

1. Introduction

The interest in waves of finite amplitude and permanent form in swirling flow arises from the study of vortex breakdown. This is the rapid change in structure which can occur in swirling flow. An example of this is vortex breakdown above a triangular wing. The reader is referred to the book by Van Dyke (1982) which contains excellent photographs of vortex breakdown.

We restrict our attention to an incompressible, inviscid fluid swirling through a right cylinder of infinite length and finite radius. The waves studied in this paper are the analogues in a swirling flow of the internal solitary and periodic gravity waves discussed in Bona, Bose, Turner (1983). For this reason we will call them solitary and periodic waves. The variational approach used here closely follows the approach used in Bona, Bose, Turner (1983). Their introduction contains a survey of the literature on internal and solitary waves. The papers of Benjamin (1962) and Pritchard (1970) discuss the theory of vortex breakdown. Pritchard also describes various experiments in rotating flows in which solitary waves were observed.

In section 2 the idealized physical model is described and the governing equations are derived. The main item of interest in this paper is the apparent difficulty arising from the singularity of the governing equations along the axis of the underlying

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cylindrical domain. The existence theory can be readily adapted to cover the singular case. The existence results for internal periodic waves are given in section 3. Both the case where the speed of propagation is specified and the case where the wave energy is specified are considered. Since the governing equations are singular along the axis of the cylindrical domain, standard elliptic theory is not applicable there. By restating the problem in a higher dimensional space, the desired regularity of solutions is achieved. This is proved in section 4 and uses an idea due to Ni (1980). Section 5 begins with a discussion of a priori bounds satisfied by the periodic solutions. These bounds are independent of the period length and are found in section 4 of Bona, Bose, Turner (1983). A method due to Amick (1984) is used to obtain the exponential decay of periodic waves from crest to trough. The final result of section 5 is the existence of internal solitary waves. They are shown to arise as the limiting forms of internal periodic waves as the period length becomes unbounded. A specific example of swirling flow is discussed in section 6 and is shown to yield governing equations which fall within the limits of our theory.

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2. The Governing Equations

To investigate the existence of internal waves in swirling flow, an idealized physical model is considered. Attention is restricted to an incompressible, inviscid fluid swirling through a right cylinder of infinite length and finite radius. Two-dimensional flows will be our domain of interest by assuming the flows to be axisymmetric. We will use cylindrical coordinates with the z-axis as the axis of the cylinder and r, the radial distance from the center of the cylinder. The radius of the cylinder is taken to be a, and thus the boundary is given by

$\{(a,\theta,z) : \theta \in [0,2\pi],\ z \in \mathbf{R}\}$.

A primary flow, $\vec{q} = (U,V,W)$, is postulated, in which the radial velocity U is zero. We also assume that, in the primary flow, the angular and axial velocities, V and W, are functions only of r.

We seek waves of permanent form whose velocity of propagation, in the direction of increasing z, is $\bar{c}$. Hence we take our coordinate system to be moving downstream at speed $\bar{c}$ so that our waveform will be stationary. $W(r)$ is replaced by $\bar{W}(r) = W(r) - \bar{c}$.

Let $\vec{q} = (u,v,w)$ denote the velocity field of a steady, incompressible flow. Incompressibility implies $\nabla\cdot\vec{q} = 0$; thus there is a Stokes stream function $\psi(r,z)$ such that

$$\frac{-\psi_z}{r} = u, \quad \frac{\psi_r}{r} = w. \tag{2.1}$$

We normalize ψ so that $\psi(0,z) = 0$. In the primary flow, with $\vec{q} = \left(0,V(r),\bar{W}(r)\right)$, we have the stream function

$$\Psi(r) = \int_0^r s\bar{W}(s)ds \tag{2.2}$$

For steady, axisymmetric flow the Euler equations in cylindrical coordinates are

$$\begin{aligned} \rho\left(uu_r + wu_z - \frac{v^2}{r}\right) &= -P_r \\ \rho\left(uv_r + wv_z + \frac{uv}{r}\right) &= 0 \\ \rho(uw_r + ww_z) &= -P_z, \end{aligned} \tag{2.3}$$

where P is the pressure and ρ is the constant density (Yih, 1979).

Denoting the vorticity of the velocity field by $\vec{\omega} = \nabla \times \vec{q} = (\xi,\eta,\zeta)$ and the total Bernoulli head by $H = \frac{P}{\rho} + \frac{1}{2}|\vec{q}|^2$, it can be shown that $\vec{q} \times \vec{\omega} = \nabla H$. Applying Kelvin's theorem to a circuit around a particular stream surface ψ = constant, we see that $rv = K(\psi)$ for some function K. From $\vec{q} \times \vec{\omega} = \nabla H$, it follows that H is also a function of ψ. We can also calculate $(\vec{\omega})_\theta = \eta = u_z - w_r = -\frac{1}{r}\left(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}\right)$. Following Benjamin (1962) or Squire (1956), we arrive at

$$r^2H'(\psi) - KK'(\psi) = \psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}. \tag{2.4}$$

Define $I = \frac{1}{2}K^2$; then

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = r^2H' - I', \tag{2.5}$$

where $'$ denotes differentiation with respect to ψ. Equation (2.5) is the fundamental equation describing our model problem. The corresponding kinematic conditions are

$$\psi(0,z) = \Psi(0) = 0 \quad \text{and} \quad \psi(a,z) = \Psi(a), \tag{2.6}$$

for $z \in \mathbb{R}$ and the asymptotic condition is

$$\psi(r,z) \to \Psi(r) \quad \text{as} \quad |z| \to \infty \tag{2.7}$$

for $r \in [0,a,]$. ψ represents a flow connected to the primary flow at infinity and for which I and H are constant on stream surfaces. H and I are not immediately known as functions of ψ, however in principle they may be determined from the primary flow. For example, if $W(r) \equiv d$, a constant, and $c = d - \bar{c}$, then by (2.2) $\Psi(r) = cr^2/2$ which may be inverted. The inverse is $R(\Psi) = \left(\frac{2\Psi}{c}\right)^{1/2}$. The "circulation" I is then expressed as a function of the stream function value Ψ by $I(\Psi) = I\big(R(\Psi)\big)$. Since I is constant on stream surfaces, its value at a point (r_0,θ,z_0), in the flow corresponding to ψ, is determined by tracing the stream surface with value $\psi(r_0,z_0)$ to infinity where I can be determined by $I(\Psi) = I\big(R(\Psi)\big)$. H is determined from $H = \frac{P}{\rho} + \frac{1}{2}|\vec{q}|^2$ and the equation of radial equilibrium in a steady cylindrical flow : $P_r = \rho v^2/r$. This is obtained from (2.3). See the example in section 6 for details on how to determine $I(\Psi)$ and $H(\Psi)$ for a specific problem.

Define the perturbed stream function $\varphi(r,z)$ by

$$\psi(r,z) = \Psi(r) + \gamma\varphi(r,z), \tag{2.8}$$

where γ is a normalizing constant. In the primary flow we have

$$\Psi_{rr} - \frac{1}{r}\Psi_r = r^2 H'(\Psi) - I'(\Psi) \quad .$$

Hence, the form of H' is

$$H'(\Psi) = \frac{1}{R^2(\Psi)}\left[\Psi_{rr}(\Psi) - \frac{1}{R(\Psi)}\Psi_r(\Psi) + I'(\Psi)\right] \quad .$$

By following stream surfaces to infinity and remembering that H is constant on stream surfaces, we find that

$$\gamma(\varphi_{rr} - \frac{1}{r}\varphi_r + \varphi_{zz}) = I'(\Psi + \gamma\varphi)\left[\frac{R^2(\Psi)}{R^2(\Psi + \gamma\varphi)} - 1\right]$$

$$+ \frac{R^2(\Psi)}{R^2(\Psi + \gamma\varphi)}\left[\Psi_{rr}(\Psi + \gamma\varphi) - \frac{\Psi_r(\Psi + \gamma\varphi)}{R(\Psi + \gamma\varphi)}\right]$$

$$- \left[\Psi_{rr}(\Psi) - \frac{\Psi_r(\Psi)}{R(\Psi)}\right]. \tag{2.9}$$

For further details on the derivation of (2.9) see Benjamin (1971), where an analogous problem in stratified flow is considered.

Equation (2.9) can be written as the nonlinear eigenvalue problem

$$\left.\begin{aligned} &-L\varphi + h(r,\varphi) = \lambda f(r,\varphi) \\ &\text{where} \\ &L = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \end{aligned}\right\} \tag{2.10}$$

The constant λ is an eigenvalue parameter proportional to c^{-2}, where c is a velocity scale for the primary flow in the travelling coordinates introduced earlier; eg. $c = \sup_{0<r<a} \overline{W}(r)$. The supplementary conditions are

$$\begin{aligned} &\varphi(0,z) = \varphi(a,z) = 0, \quad z \in R, \\ &\varphi(r,z) \to 0 \quad \text{as} \quad |z| \to \infty,\ r \in [0,a]. \end{aligned} \tag{2.11}$$

In this paper it will be shown that, under suitable conditions on f and h, the boundary value problem (2.10, 2.11) has solitary wave solutions. By a "solitary wave" solution we mean a solution φ which is even in z, monotone for $z > 0$, and rapidly convergent to zero as $|z| \to \infty$. In the course of obtaining solitary wave solutions it will be shown that there are solutions φ of (2.10) which vanish at $r = 0$ and $r = a$, which are even, periodic functions of z with period $2k$, and which are monotone for $z \in [0,k]$. These will be called periodic waves.

3. Periodic Waves

In this section we prove the existence of periodic solutions (λ,φ) of the boundary value problem

$$
\left.\begin{aligned}
-L\varphi(r,z) + h\big(r,\varphi(r,z)\big) &= \lambda f\big(r,\varphi(r,z)\big) \quad \text{in} \quad \Omega, \\
\varphi(0,z) = \varphi(a,z) &= 0,
\end{aligned}\right\} \qquad (P)\ (3.1)
$$

where L is given by (2.10) and $\Omega = \{(r,z) : r \in (0,a),\ z \in \mathbb{R}\}$.

We make the following hypotheses on f and h.

Hypothesis (H):

(HI): The function f has the form

$$
f(r,t) = \begin{cases} t\, f_0(r) + f_1(r,t) & r \in [0,a], \quad t > 0 \\ -\, f(r,\, -\, t) & r \in [0,a], \quad t < 0 \end{cases}
$$

with f_0, f_1 Hölder continuous on bounded r sets and f_1 Lipschitz continuous in t on bounded sets. We further assume that $f_0 > 0$ for $r \in (0,a)$, $f_1(r,t) = o(t)$, uniformly for $r \in [0,a]$, as $t \to 0$; and that there exist constants $m,n > 1$ and $\alpha, d > 0$ such that for $t > 0$

$$\alpha t^m \le f_1(r,t) \le d(1 + t^n).$$

(HII): The function h has the form

$$
h(r,t) = \begin{cases} t\, h_0(r) + h_1(r,t) & r \in [0,a], \quad t > 0 \\ -\, h(r,\, -\, t) & r \in [0,a], \quad t < 0 \end{cases}
$$

with h_0, h_1 Hölder continuous on bounded r sets and h_1 Lipschitz continuous in t on bounded sets. Let e_0 be the lowest eigenvalue of $-\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr}$ in $[0,a]$ with Dirichlet data. A lower bound for e_0 is given in lemma 3.4. For h we further assume that $\dfrac{\partial h}{\partial t}$ is continuous, $\dfrac{\partial h}{\partial t} > e$ where $e > -e_0$, and that there exist constants $\sigma > 0$ and σ' such that for $t > 0$,

$$\sigma' t^m \le h_1(r,t) \le \sigma t^m,$$

and

$$|h_1(r,t)| \le d(1 + t^n),$$

where $m,n,$ and d are the constants in (HI). The first inequality implies $h_1(r,t) = o(t)$, uniformly for $r \in [0,a]$, as $t \to 0$.

Finally we define the functions

$$F(r,t) = 2\int_0^t f(r,s)ds, \quad F_1(r,t) = 2\int_0^t f_1(r,s)ds$$
$$H(r,t) = 2\int_0^t h(r,s)ds, \quad H_1(r,t) = 2\int_0^t h_1(r,s)ds. \tag{3.2}$$

(HIII): The last hypothesis is,

For each $\lambda > \sigma/\alpha$, where α and σ appear in (HI) and (HII), there is a $\theta \in (0,1)$ such that

$$\lambda F_1(r,t) - H_1(r,t) \le \theta\big(\lambda f_1(r,t) - h_1(r,t)\big)t.$$

We now want to find solutions of (P) which are periodic in z with period $2k$. Let

$$\Omega_{k,2} = \{(r,z) : r \in (0,a),\ z \in (-k,k)\}. \tag{3.3}$$

The subscript 2 here is to distinguish the set from higher dimensional analogues to be introduced later. The problem (P) can be formulated in two ways. The first is a constrained problem (PC):

$$\left.\begin{array}{l} \text{solve (P)} \\ \text{subject to } \displaystyle\int_{\Omega_{k,2}} \frac{1}{r}\left\{|\nabla\varphi|^2 + H(r,\varphi)\right\} drdz = R^2 \\ \text{where } R > 0 \text{ is a given constant.} \end{array}\right\} \quad \text{(PC)} \tag{3.4}$$

The second is a free problem (PF):

$$\left.\begin{array}{l} \text{solve (P)} \\ \text{where } \lambda \text{ is a given constant.} \end{array}\right\} \quad \text{(PF)} \tag{3.5}$$

(PC) corresponds to specifying the "energy" of a wave, while (PF) corresponds to specifying its velocity.

The analysis of both problems is based on variational techniques and the theory of rearrangements of functions. In each problem a critical point φ of a suitable

functional will be found and shown to be a solution of a weak formulation of the problem. In addition, φ can be taken to be even in z, nonnegative, and nonincreasing on $z \in [0,k]$ for each r. The following definition is used.

<u>Definition 3.1</u>: Let $\varphi = \varphi(r,z)$ be continuous on $\Omega_{k,2}$ and periodic in z with period $2k$. For each r let $\mu(\varphi,\gamma,r)$ denote the Lebesque measure of the set $\{z : \varphi(r,z) > \gamma\}$. A function $\hat{\varphi}$ which is even in z, nonnegative, nonincreasing on $z \in [0,k]$ for each r and satisfies $\mu(\hat{\varphi},\gamma,r) = \mu(|\varphi|,\gamma,r)$ is called the symmetrization of φ. If $\varphi = \hat{\varphi}$ we call it symmetrized.

The analysis will be carried out in the Hilbert space $H_k = H_k\big((0,a) \times R\big)$ defined as follows: let C_k^∞ denote the C^∞ functions which have support where $r \in (0,a)$ and which are periodic in z with period $2k$. Define

$$\|u\|_k^2 = \int_{\Omega_{k,2}} \frac{1}{r} \left\{|\nabla u|^2 + h_0(r)u^2\right\} drdz. \tag{3.6}$$

The Poincare inequality (see lemma 3.4)

$$\int_{\Omega_{k,2}} \frac{1}{r}|\nabla u|^2 drdz \geq e_0 \int_{\Omega_{k,2}} \frac{1}{r} u^2 \, drdz \tag{3.7}$$

together with $h_0(r) \geq e > -e_0$, a consequence of (HII), show that $\|\ \|_k$ is a norm. Let H_k be the completion of C_k^∞ in this norm; the space H_k is a Hilbert space with inner product

$$(u,v)_k = \int_{\Omega_{k,2}} \frac{1}{r} \left\{\nabla u \cdot \nabla v + h_0(r)uv\right\} drdz. \tag{3.8}$$

If ℓ is a continuous linear functional on H_k, its value at u is denoted by $\langle \ell,u \rangle$.

The use of symmetrized functions is not needed in showing the existence of periodic solutions, however it is very important in getting the estimates in section 5 which are used to show that a solitary wave can be obtained as the limit of periodic waves with increasing period. For the construction of $\hat{\varphi}$, using piecewise linear functions and an approximation process, and the properties of $\hat{\varphi}$, the reader is referred to the appendix

of Bona, Bose, Turner (1983). A version of the following lemma can be found in Polya, Szego (1951).

Lemma 3.2: Suppose $G(r,u)$ is even in u and continuous for $(r,u) \in [0,a] \times \mathbb{R}$. Then, for u piecewise linear and $u_z \neq 0$ a.e.,

$$\int_{\Omega_{k,2}} G(r,u)\,drdz = \int_{\Omega_{k,2}} G(r,\hat{u})\,drdz. \tag{3.9}$$

Lemma 3.3: Suppose $p(r)$ is positive and measurable on $[0,a]$ and u is piecewise linear and $2k$ periodic in z with $u_z \neq 0$ a.e. Then

$$\int_{\Omega_{k,2}} p(r)\,|\nabla u|^2 drdz > \int_{\Omega_{k,2}} p(r)\,|\nabla \hat{u}|^2 drdz. \tag{3.10}$$

Proof: The case $p \equiv 1$ is found in Polya, Szego (1951). By continuity we may assume $p \in C^2([0,a])$ and $u \in C_k^\infty$. Let $q^2 = p$ with $q(r) > 0$. Consider

$$|\nabla(qu)|^2 = q^2|\nabla u|^2 + q_r^2u^2 + 2qq_r uu_r.$$

Then

$$\int_{\Omega_{k,2}} q_r q(u^2)_r\,drdz = -\int_{\Omega_{k,2}} (q_{rr}qu^2 + q_r^2u^2)\,drdz.$$

so

$$\int_{\Omega_{k,2}} p|\nabla u|^2 drdz = \int_{\Omega_{k,2}} |q\nabla u|^2 drdz$$

$$= \int_{\Omega_{k,2}} |\nabla(qu)^2|\ drdz - \int_{\Omega_{k,2}} (q_r^2u^2 + 2q_r qu_r u)\,drdz$$

$$= \int_{\Omega_{k,2}} |\nabla(qu)|^2 drdz + \int_{\Omega_{k,2}} q_{rr}qu^2\,drdz.$$

Since the symmetrization is in z and since q is a positive function of r, $\widehat{qu} = q\hat{u}$. By lemma 3.2, with $G(r,u) = q_{rr}(r)q(r)u^2(r,z)$,

$$\int_{\Omega_{k,2}} q_{rr}qu^2\,drdz = \int_{\Omega_{k,2}} q_{rr}q\hat{u}^2\,drdz.$$

Finally,

$$\int_{\Omega_{k,2}} p|\nabla u|^2 drdz = \int_{\Omega_{k,2}} |\nabla(qu)|^2 drdz + \int_{\Omega_{k,2}} q_{rr} qu^2 drdz$$

$$\geq \int_{\Omega_{k,2}} |\nabla(\widehat{qu})|^2 drdz + \int_{\Omega_{k,2}} q_{rr} q\hat{u}^2 drdz$$

$$= \int_{\Omega_{k,2}} |\nabla(q\hat{u})|^2 drdz + \int_{\Omega_{k,2}} q_{rr} q\hat{u}^2 drdz$$

$$= \int_{\Omega_{k,2}} p|\nabla\hat{u}|^2 drdz.$$

<u>Lemma 3.4</u>: For $u \in H_k$ the constant e_0 in the Poincare inequality (3.7) satisfies

$$e_0 \geq \frac{4\pi^2+3}{4a^2}.$$

<u>Proof</u>: By continuity we assume $u \in C_k^\infty$. Using the calculation in the proof of lemma 3.3 with $q = 1/\sqrt{r}$,

$$\int_{\Omega_{k,2}} \frac{1}{r}|\nabla u|^2 drdz = \int_{\Omega_{k,2}} |\nabla(\frac{u}{\sqrt{r}})|^2 drdz + \int_{\Omega_{k,2}} \frac{3u^2}{4r^3} drdz$$

$$\geq \frac{\pi^2}{a^2}\int_{\Omega_{k,2}} \frac{u^2}{r} drdz + \frac{3}{4a^2}\int_{\Omega_{k,2}} \frac{u^2}{r} drdz$$

$$= \frac{4\pi^2+3}{4a^2}\int_{\Omega_{k,2}} \frac{1}{r} u^2 drdz.$$

Along with H_k, the spaces C^j, $C^{j,\omega}$, $L^p(\Omega)$, and $W^{j,p}(\Omega)$ will be used for various domains. The reader is referred to Adams (1975) or Gilbarg, Trudinger (1983). Here and in what follows C denotes a positive constant independent of k. The constant changes from one inequality to another.

<u>Lemma 3.5</u>: The following relation holds,

$$H_k \subset W^{1,2}(\Omega_{k,2}) \subset L^p(\Omega_{k,2}), \qquad p \in [2,\infty).$$

<u>Proof</u>: From the Poincare inequality and Sobolev imbedding theorem,

$$\begin{aligned}
\|u\|_{L^p(\Omega_{k,2})} &= \left(\int_{\Omega_{k,2}} |u|^p drdz\right)^{1/p} \qquad 2 \le p < \infty \\
&\le C \left(\int_{\Omega_{k,2}} \{|\nabla u|^2 + u^2\} drdz\right)^{1/2} \\
&= C\,\|u\|_{W^{1,2}(\Omega_{k,2})} \\
&\le C\sqrt{a}\left(\int_{\Omega_{k,2}} \frac{1}{r}\{|\nabla u|^2 + u^2\} drdz\right)^{1/2} \\
&\le C\left(\int_{\Omega_{k,2}} \frac{1}{r}\{|\nabla u|^2 + h_0 u^2\} drdz\right)^{1/2} \\
&= C\|u\|_k .
\end{aligned}$$

A further consequence of the Poincare inequality is, for $u \in H_k$,

$$\|u\|_k^2 \le C \int_{\Omega_{k,2}} \frac{1}{r} |\nabla u|^2 drdz. \tag{3.11}$$

For convenience we will often use

$$\int u = \int_{\Omega_{k,2}} u(r,z) drdz. \tag{3.12}$$

If $u \in H_k$, $Lu/\sqrt{r} \in L^2(\Omega_{k,2})$ and $v \in H_k$, then

$$\begin{aligned}
(u,v)_k &= \int \frac{1}{r} \{\nabla u \cdot \nabla v + h_0(r)uv\} \\
&= \int \frac{1}{r} \{-Lu + h_0(r)u\} v,
\end{aligned} \tag{3.13}$$

where the boundary terms cancel on $z = \pm k$ due to periodicity. Thus we define a weak periodic solution of (P) as a pair (λ,φ), $\varphi \in H_k$, satisfying

$$(\varphi,v)_k + \int \frac{1}{r} h_1(r,\varphi)v = \lambda \int \frac{1}{r} f(r,\varphi)v \qquad (3.14)$$

for all $v \in H_k$. The integrals in (3.14) exist by virtue of (HI), (HII), lemma 3.5, and lemma 3.6.

<u>Lemma 3.6</u>: For $u \in H_k$ and $p > 2$,

$$\int \frac{1}{r}|u|^p < C \|u\|_k^p. \qquad (3.15)$$

<u>Proof</u>: Let $q = 1/p$, then

$$\int \frac{1}{r}|u|^p = \int \left|\frac{u}{r^q}\right|^p = \left|\frac{u}{r^q}\right|^p_{L^p(\Omega_{k,2})}.$$

From the proof of lemma 3.5,

$$\left|\frac{u}{r^q}\right|_{L^p(\Omega_{k,2})} < C \left(\int \left\{\left|\nabla\left(\frac{u}{r^q}\right)\right|^2 + \frac{u^2}{r^{2q}}\right\}\right)^{1/2}.$$

Since $p > 2$, $q < 1/2$ and $r^{-2q} < C r^{-1}$.

Considering now the gradient term and integrating by parts as in the proofs of lemmas 3.3 and 3.4,

$$\int\left|\nabla\left(\frac{u}{r^q}\right)\right|^2 = \int \frac{|\nabla u|^2}{r^{2q}} - (q^2+q)\int \frac{u^2}{r^{2q+2}}$$

$$< C\int \frac{|\nabla u|^2}{r} - (q^2+q)\int \frac{u^2}{r^{2q+2}}$$

$$< C\int \frac{|\nabla u|^2}{r}.$$

Hence

$$\int \left\{\left|\nabla\left(\frac{u}{r^q}\right)\right|^2 + \frac{u^2}{r^{2q}}\right\} < C\int\left\{\frac{|\nabla u|^2}{r} + \frac{u^2}{r}\right\} < C\|u\|_k^2$$

and the result follows.

First we consider the problem (PC). For $u \in H_k$ introduce

$$\left.\begin{aligned} &\Lambda(u) = \int \frac{1}{r}\{|\nabla u|^2 + H(r,u)\} \\ &J(u) = \int \frac{1}{r} F(r,u) \\ &\text{where } F \text{ and } H \text{ are defined in (3.2) and for } R > 0, \text{ let} \\ &S(R) = \{u \in H_k : \Lambda(u) = R^2\}. \end{aligned}\right\} \qquad (3.16)$$

The assumptions on f and h, along with lemma 3.6, guarantee that the functionals Λ and J are defined on H_k. One calls φ a critical point of J on $S(R)$ if the derivative of J is zero in directions tangent to $S(R)$, ie. $J'(\varphi)$ is parallel to $\Lambda'(\varphi)$. A consequence of this is (3.14).

Theorem 3.7: Let f and h satisfy (H). Then for each $k > 0$ (PC) has a solution (λ_k, φ_k), $\varphi_k \in H_k \cap C^{2,\omega}(\Omega_{k,2})$, such that

(1) $J(\varphi_k) = \sup_{u \in S(R)} J(u)$,

(2) $\lambda_k > 0$ and $\varphi_k > 0$ in $\Omega_{k,2}$,

(3) $\varphi_k = \hat{\varphi}_k$ (cf. definition 3.1).

Proof: Apart from regularity, the proof follows the lines of theorem 3.2 in Bona, Bose, Turner (1983). Due to the $1/r$ singularity in the operator L (2.10), L^p elliptic theory from Agmon, Douglis, Nirenberg (1959) is applicable only to regions $\Omega' = \{(r,z) \in \Omega_{k,2} : r > \epsilon > 0\}$. Since φ is a weak solution of an elliptic equation, $\varphi \in W^{2,p}(\Omega')$, $p \in [2,\infty)$. By the Sobolev imbedding theorem and the Schauder theory, see Gilbarg, Trudinger (1983), $\varphi \in C^{2,\omega}(\overline{\Omega}')$.

Remark: In section 4 it is shown that $\varphi \in C^{2,\omega}(\overline{\Omega}_{k,2})$, ie. that φ is regular at $r = 0$.

Now we turn to (PF). Let μ be the smallest eigenvalue of the linear problem

$$-Lu + h_0(r)u = \lambda f_0(r)u,$$
$$u \in H_k. \tag{3.17}$$

Using separation of variables we see that μ has a corresponding eigenfunction $\nu(r)$ associated with the lowest eigenvalue of

$$-\nu_{rr} + \frac{1}{r}\nu_r + h_0(r)\nu = \lambda f_0(r)\nu,$$
$$\nu(0) = \nu(a) = 0. \tag{3.18}$$

In fact the equation is of limit point type at $r = 0$ with respect to L^2 with the weight $1/r$, so the condition $\nu(0) = 0$ is superfluous. We can also use the change of variables $y = r^2/2$ to get

$$-v_{yy} + \frac{\tilde{h}_0(y)v}{2y} = \frac{\lambda\tilde{f}_0(y)v}{2y},$$

where $v(y) = v(r^2/2) = \nu(r)$. The existence and positivity of ν follow from arguments like those used in the proof of theorem 3.7. We normalize ν such that

$$\int_0^a \frac{1}{r}\{\nu_r^2 + h_0\nu^2\}dr = 1. \tag{3.19}$$

For λ fixed define a functional M on H_k by

$$M(u) = \|u\|_k^2 + \int \frac{1}{r}\{H_1(r,u) - \lambda F(r,u)\}. \tag{3.20}$$

$\|M'(u)\|$ denotes the norm of the derivative $M'(u)$ as a functional on H_k. A critical point φ of M, ie. $\langle M'(\varphi),v\rangle = 0$ for all $v \in H_k$, is a weak solution of (PF) by (3.14). Now we show (PF) has a symmetrized solution by first stating a technical result from Bona, Bose, Turner (1983).

<u>Proposition 3.8</u>: Fix $k > 0$ and let M be a continuously differentiable functional on H_k satisfying:

(1) There are constants $s, \bar{s} > 0$ such that $M(u) \geq s$ for $\|u\|_k = \bar{s}$.

(2) $M(0) = 0$ and there exists a function w with $\|w\|_k > \bar{s}$ such that $M(w) < 0$.

(3) For each $\beta > 0$ if $\{u_i\}_{i=1}^{\infty}$ is a sequence satisifying $\beta < M(u_i) < 1/\beta$, for all i, and $\|M'(u_i)\| \to 0$ as $i \to \infty$, then there is a subsequence $\{u_{i_m}\}_{m=1}^{\infty}$ of $\{u_i\}_{i=1}^{\infty}$ converging strongly in H_k as $m \to \infty$.

Let

$$\Gamma = \{\gamma \in C([0,1],H_k) : \gamma(0) = 0,\ \gamma(1) = w\}$$

and define

$$b_\gamma = b_\gamma(M) = \max_{u \in \gamma([0,1])} \quad ,$$

$$b = \inf_{\gamma \in \Gamma} b\gamma. \tag{3.21}$$

Suppose γ_n, $n \in \mathbb{Z}^+$, is a sequence of paths in Γ such that $b_{\gamma_n} < b + 1/n$. Then there is a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{Z}^+$ and functions $u_{n_j} \in \gamma_{n_j}([0,1])$ such that u_{n_j} converges strongly to u as $j \to \infty$, $M(u) = b$, and $M'(u) = 0$.

<u>Theorem 3.9</u>: Let f and h satisfy (H) with α and σ as defined. Let μ be the lowest eigenvalue of (3.17). If $\lambda \in (\sigma/\alpha, \mu)$, then for each $k > 0$ (PF) has a solution $\varphi_k \in H_k \cap C^{2,\omega}(\Omega_{k,2})$ with

(1) $M(\varphi_k) = \inf_{\gamma \in \Gamma} b_\gamma(M)$,

(2) $\varphi_k > 0$ in $\Omega_{k,2}$,

(3) $\varphi_k = \hat{\varphi}_k$.

<u>Proof</u>: The proof parallels that of theorem 3.4 in Bona, Bose, Turner (1983). One constructs a sequence of minimizing paths consisting of symmetrized functions. Proposition 3.8 guarantees that the critical point will be symmetrized.

<u>Remark</u>: Again we postpone the regularity of φ at $r = 0$ until section 4.

4. Regularity At $r = 0$

In this section we obtain regularity up to $r = 0$ where the equation

$$-(\varphi_{rr} - \frac{1}{r}\varphi_r + \varphi_{zz}) + h(r,\varphi) = \lambda f(r,\varphi) \tag{4.1}$$

is singular and standard elliptic theory breaks down. To extend the regularity of φ from $C^{2,\omega}(\Omega_{k,2})$ to $C^{2,\omega}(\bar{\Omega}_{k,2})$ we use an approach due to Ni (1980), which involves restating the problem in $\mathbb{R}^5$.

Set $\varphi = r^2 g$. We will fix k for the entire section and suppress it. We then have

$$\begin{aligned} L\varphi &= \varphi_{rr} - \frac{1}{r}\varphi_r + \varphi_{zz} \\ &= (r^2 g)_{rr} - \frac{1}{r}(r^2 g)_r + (r^2 g)_{zz} \\ &= r^2 g_{rr} + 3rg_r + r^2 g_{zz}. \end{aligned}$$

Hence (4.1) becomes

$$-\left(g_{rr} + \frac{3}{r}g_r + g_{zz}\right) + \frac{h(r,r^2 g)}{r^2} = \frac{\lambda f(r,r^2 g)}{r^2} \tag{4.2}$$

with boundary conditions

$$g(a,z) = 0, \quad g(r,k) = g(r,-k).$$

The operator which has replaced L is the Laplacian in cylindrical coordinates in $\mathbb{R}^5$ as applied to functions with cylindrical symmetry. Since the nonlinear terms f and h have polynomial growth in $(r^2 g)$, division by r^2 is not a problem.

We define g in $\mathbb{R}^5$ to be axially symmetric with respect to the axis $r = 0$ (z-axis) such that $r^2 g\big|_{\Omega_{k,2}} = \varphi$. $\Omega_{k,5}$ is the domain in $\mathbb{R}^5$ generated by rotating $\Omega_{k,2}$ around the axis $r = 0$. Note that $r = 0$ is not included in $\Omega_{k,5}$.

<u>Lemma 4.1</u>: g is a classical solution of (4.2) in the open set $\Omega_{k,5}$.

<u>Proof</u>: This follows immediately from knowing $\varphi = r^2 g\big|_{\Omega_{k,2}}$ is a classical solution of (3.1) for $r > 0$.

We will use $L^p(\Omega_{k,5})$ and $W^{j,p}(\Omega_{k,5})$ to denote the L^p and $W^{j,p}$ spaces in $\mathbb{R}^5$. We omit the angular factor of the Jacobian, $r^3 \sin^2\theta_1 \sin\theta_2 \, dr d\theta_1 \, d\theta_2 d\theta_3 dz$, in the integrals for $\Omega_{k,5}$.

Lemma 4.2: For $\varphi \in H_k$ and $r^2 g|_{\Omega_{k,2}} = \varphi$,

$$\|g\|_k^2 = \int_{\Omega_{k,5}} \{|\nabla g|^2 + h_0 g^2\} r^3 drdz = \|\varphi\|_k^2.$$

Proof: $|\nabla(r^2 g)|^2 = r^4|\nabla g|^2 + 4r^3 g g_r + 4r^2 g^2$

and

$$\int_{\Omega_{k,5}} r^2 \left(\frac{g^2}{2}\right)_r drdz = -\int_{\Omega_{k,5}} r g^2 drdz,$$

so

$$\int_{\Omega_{k,2}} \frac{1}{r} |\nabla(r^2 g)|^2 drdz = \int_{\Omega_{k,5}} r^3 |\nabla g|^2 drdz.$$

Therefore,

$$\begin{aligned}\|\varphi\|_k^2 &= \int_{\Omega_{k,2}} \frac{1}{r} \{|\nabla\varphi|^2 + h_0(r)\varphi^2\}\, drdz \\ &= \int_{\Omega_{k,5}} \{|\nabla g|^2 r^3 + \frac{h_0}{r}(r^2 g)^2\}\, drdz \\ &= \int_{\Omega_{k,5}} \{|\nabla g|^2 + h_0 g^2\}\, r^3 drdz \\ &= \|g\|_k^2.\end{aligned}$$

Theorem 4.3: g is a weak solution of

$$-\Delta g = \frac{\lambda f(r,r^2 g) - h(r,r^2 g)}{r^2} \quad \text{in } B_{k,5} \tag{4.3}$$

with $g(a,z) = 0$ for $z \in (-k,k)$ and g periodic in z with period $2k$. $B_{k,5} = \{(0,z) : z \in (-k,k)\} \cup \Omega_{k,5}$.

Proof: The reader is referred to Ni (1980), where it is shown Δg does not produce a distribution supported at $r = 0$.

__Lemma 4.4__: Suppose $u = u(r,z) \in W^{1,2}(B_{k,5})$ with $u(a,z) = 0$ for $z \in (-k,k)$, $q > 1$, and $\beta > \frac{3q}{2}$. Then

$$\Big(\int_{B_{k,5}} |u|^q r^\beta drdz\Big)^{1/q} < C\Big(\int_{B_{k,5}} |\nabla u|^2 r^3 drdz\Big)^{1/2}. \tag{4.4}$$

__Proof__: By continuity we assume $u \in C^\infty(B_{k,5})$. The Sobolev inequality in $\mathbb{R}^2$ is

$$\Big(\int_{\Omega_{k,2}} |ur^{\beta/q}|^q drdz\Big)^{1/q} < C\Big(\int_{\Omega_{k,2}} |\nabla(ur^{\beta/q})|^2 drdz\Big)^{1/2}.$$

Consider

$$|\nabla(ur^{\beta/q})|^2 = |\nabla u|^2 r^{2\beta/q} + \frac{\beta}{q}(u^2)_r\, r^{(2\beta/q)-1}$$

$$+ \frac{\beta^2}{q^2} u^2\, r^{(2\beta/q)-2}.$$

Since

$$\int_{\Omega_{k,2}} \frac{\beta}{q}(u^2)_r r^{(2\beta/q)-1} drdz = \frac{\beta}{q}\Big(1 - \frac{2\beta}{q}\Big)\int_{\Omega_{k,2}} u^2 r^{(2\beta/q)-2} drdz,$$

we have

$$\int_{\Omega_{k,2}} |\nabla(ur^{\beta/q})|^2 drdz = \int_{\Omega_{k,2}} |\nabla u|^2 r^{2\beta/q} drdz + \frac{\beta}{q}\Big(1 - \frac{\beta}{q}\Big)\int_{\Omega_{k,2}} u^2 r^{(2\beta/q)-2} drdz$$

$$< \int_{\Omega_{k,2}} |\nabla u|^2 r^{2\beta/q} drdz$$

$$< \sup_{[0,a]} r^{(2\beta/q)-3} \int_{\Omega_{k,2}} |\nabla u|^2 r^3 drdz$$

$$< C\int_{\Omega_{k,2}} |\nabla u|^2 r^3 drdz.$$

The result follows immediately.

Now we state the main result of this section.

__Theorem 4.5__: $g \in C^{2,\omega}(\bar{\Omega}_{k,5})$ or equivalently $\varphi/r^2 \in C^{2,\omega}(\bar{\Omega}_{k,2})$.

Proof: By (H), $|\lambda f(r,t) - h(r,t)| \le C(|t| + |t|^n)$, and so the right hand side of (4.3) is bounded:

$$\left|\frac{\lambda f(r,r^2g) - h(r,r^2g)}{r^2}\right| \le C(|g| + r^{2n-2}|g|^n).$$

Standard elliptic theory is applicable to (4.3) since we are now dealing with the Laplacian. Lemma 4.2 shows $g \in W^{1,2}(B_{k,5})$. For $n < 7/3$, a bootstrapping argument and the crude estimate $C(|g| + r^{2n-2}|g|^n) \le C(|g| + |g|^n)$ give us $g \in W^{2,p}(B_{k,5})$ for all $p > 1$. Invoking lemma 4.4 with $p = 5/2$, $q = np$, and $\beta = (2n-2)p + 3$,

$$\int_{B_{k,5}} (r^{2n-2}|g|^n)^p\, r^3 dr dz < \infty$$

if $(2n-2)\frac{5}{2} + 3 > \frac{3}{2} \cdot \frac{5n}{2}$, ie. $n > 8/5$. Hence, for $n > 8/5$, the right hand side of (4.3) is in $L^{5/2}(B_{k,5})$. Thus, by elliptic theory, $g \in W^{2,5/2}(B_{k,5})$ for all $n > 1$. It follows from the Sobolev imbedding theorem that $g \in L^p(B_{k,5})$ for $p \in [1,\infty)$. Since, by elliptic theory, $g \in W^{2,p}(B_{k,5})$ for $p \in [1,\infty)$ a further application of the Sobolev imbedding theorem gives $g \in C^{1,\omega}(B_{k,5})$. By our assumptions on f and h, $\frac{\lambda f(r,r^2g) - h(r,r^2g)}{r^2} \in C^{\omega}(B_{k,5})$. The Schauder theory gives $g \in C^{2,\omega}(\overline{B}_{k,5})$. Our result follows from noting that $\overline{B}_{k,5} = \overline{\Omega}_{k,5}$.

5. Solitary Waves

As in Bona, Bose, Turner (1983) it can be shown that $\lambda_k < \mu$ and $\|\varphi_k\|_k$ is bounded independent of k for both (PF) and (PC). The additional assumptions of $m < 5$ and $2\sigma/\alpha < \mu$ are needed here, where m, σ, and α appear in (H). Corollary 4.5 in Bona, Bose, Turner (1983) gives a bound on $\|g_k\|_{C^{2,\omega}}$ independent of k. Since our solutions of (PF) and (PC) are symmetrized in z, for k sufficiently large we can obtain the following crude decay estimate

$$g_k(r,z) \le \frac{C\|\varphi_k\|_k \left(1 + \|\varphi_k\|_k^{n-1}\right)^{1/3}}{|z|^{1/3}} \tag{5.1}$$

for $|z| < k$. See lemma 4.9 of Bona, Bose, Turner (1983) for details.

Using this decay rate we can obtain exponential decay rates for φ_k and $\nabla\varphi_k$. The approach we use is due to Amick (1984). He used it to get exponential decay for solitary waves, but it is easily adapted to periodic waves.

Lemma 5.1: Let (λ_k,φ_k), $\varphi_k \in H_k$, be a solution of (P) from either theorem 3.7 or theorem 3.9. Suppose that for some $q > 1$,

$$|\lambda_k f_1(r,t) - h_1(r,t)| \le C_0 t^q$$

for $|t| \le 1$ and $r \in [0,a]$. Then for $k > k_0$, $|z| \le k$,

$$\varphi_k(r,z) \le Ce^{-\beta|z|} \tag{5.2}$$

and

$$|\nabla\varphi_k(r,z)| \le C'e^{-\beta|z|} \tag{5.3}$$

where C and C' depend on a,q,β, and $\|g_k\|_{C^{2,\omega}}$. $\beta = \frac{1}{2}\left[(e + e_0)\left(1 - \frac{\lambda_k}{\mu}\right)\right]^{1/2}$.

Proof: As noted above, $\lambda_k < \mu$. We now suppress the subscript k. Let $Q(r,g) = \dfrac{\lambda f_1(r,r^2g) - h_1(r,r^2g)}{r^2}$. Again we use $\varphi = r^2g|_{\Omega_{k,2}}$ where g satisfies (4.2), and

$$|Q(r,g)| \le C_1g^q \quad \text{if} \quad r^2g \le 1 .$$

For $r \in [0,a]$, $z \in [-k,k]$, (5.1) gives $g(r,z) \le \dfrac{C_2}{|z|^{1/3}}$. For k_0 large enough $r^2g \le 1$ on $[0,a] \times [k_0,k]$. We can rewrite (4.2) as

$$-\nabla\cdot(r^3\nabla g) + h_0gr^3 = \lambda f_0gr^3 + Q(r,g)r^3 .$$

Multiplying by g and integrating over $(0,a) \times (z,k)$ gives, after an integration by parts,

$$\begin{aligned}\int_z^k\int_0^a \{|\nabla g|^2 + h_0g^2\}r^3drd\zeta - \int_z^k\int_0^a \lambda f_0g^2r^3drd\zeta \\ = \int_0^a g(r,k)g_z(r,k)r^3dr - \int_0^a g(r,z)g_k(r,z)r^3dr + \int_z^k\int_0^a Q(r,g)gr^3drd\zeta\end{aligned} \tag{5.4}$$

μ is the smallest eigenvalue of the linear problem (3.17) and $g_z(r,k) < 0$ since $g = \hat{g}$, so

$$\left(1-\frac{\lambda}{\mu}\right)\int_z^k\int_0^a \{|\nabla g|^2 + h_0 g^2\} r^3 dr d\zeta < -\int_0^a g(r,z)g_z(r,z)r^3 dr + \int_z^k\int_0^a Q(r,g)gr^3 dr d\zeta. \tag{5.5}$$

From the proof of lemma 4.2 and our Poincare inequality (3.7)

$$\int\int|\nabla g|^2 r^3 dr d\zeta > e_0 \int\int g^2 r^3 dr d\zeta.$$

Using this and the hypothesis on $|Q|$, (5.5) becomes, for $z \in (k_0,k)$,

$$\left(1-\frac{\lambda}{\mu}\right)\int_z^k\int_0^a (e_0 + h_0)g^2 r^3 dr d\zeta < C_1\int_z^k\int_0^a g^{q+1}r^3 dr d\zeta - \int_0^a g(r,z)g_z(r,z)r^3 dr. \tag{5.6}$$

From (HII) $h_0(r) > e > -e_0$, so let $C_3 = (e_0 + e)\left(1-\frac{\lambda}{\mu}\right)$. Then

$$C_3\int_z^k\int_0^a g^2r^3 dr d\zeta + \int_0^a g(r,z)g_z(r,z)r^3 dr < C_1 \max_{z<k_0} |g(r,z)|^{q-1}\int_z^k\int_0^a g^2 r^3 dr d\zeta,$$

for $z \in (k_0,k)$. Choose k_0 such that

$$C_1 \max_{z<k_0} |g(r,z)|^{q-1} < C_1\left(\frac{C_2}{k_0^{1/3}}\right)^{q-1} < \frac{C_3}{2}.$$

Now for $z \in (k_0,k)$ we have

$$\frac{C_3}{2}\int_z^k\int_0^a g^2 r^3 dr d\zeta + \int_0^a g(r,z)g_z(r,z)r^3 dr < 0 \tag{5.7}$$

As in Amick (1984), it follows that

$$\int_z^k\int_0^a g^2 r^3 dr d\zeta < C_4 e^{-\sqrt{C_3}\, z}$$

for $z \in (k_0,k)$. Since $\|g\|_{C^{2,\omega}}$ is bounded, we obtain

$$g(r,z) < C_5 e^{-\beta|z|}$$

on $[0,a] \times [-k,k]$, $k > k_0$, where $\beta = \sqrt{C_3}/2$.

Elliptic estimates (eg. L^p estimates and the Sobolev inequalities) give local L^∞ bounds for ∇g in terms of local L^∞ bounds for g. It follows that

$$|\nabla g(r,z)| \le C_6 e^{-\beta|z|}.$$

We now show the existence of solitary waves for (PF) and (PC). They arise as the limit of periodic waves as the period $k \to \infty$.

Theorem 5.2: Assume f and h satisfy (H) and let α, σ, and m be the constants defined there. Let μ be the lowest eigenvalue of the linear problem (3.17). Suppose also that $m < 5$, $2\sigma/\alpha < \mu$, and $|\lambda f_1(r,t) - h_1(r,t)| \le Ct^q$, for some $q > 1$, on $[0,a] \times [-1,1]$. Then

(PC): Let $R > 0$ be given. For each $k > 0$ let (λ_k, φ_k) be a solution of (PC) from theorem 3.7 with $\Lambda(\varphi_k) = R^2$. Then there exists an increasing sequence of half periods $k(j) \to \infty$, $j \in \mathbb{Z}^+$, and a solution (λ, φ) of (P) satisfying

(1) $\varphi > 0$ and $\varphi = \hat{\varphi}$ on Ω,

(2) $\Lambda(\varphi) = R^2$,

(3) $\|\varphi\|_{C^{2,\omega}} \le C(R+R^n)^\omega$,

(4) $|\varphi|$, $|\nabla\varphi| \le Ce^{-\beta|z|}$, $\beta = \frac{1}{2}[(e + e_0)(1 - \lambda/\mu)]^{1/2}$

(5) $\lim_{j \to \infty} \lambda_{k(j)} = \lambda$, $\lambda \in (0,\mu)$,

(6) $\lim_{j \to \infty} \varphi_{k(j)} = \varphi$ uniformly in C^2 on bounded subsets of Ω.

(PF): Let $\lambda \in (\sigma/\alpha, \mu)$ be given. For each $k > 0$ let (λ, φ_k) be a solution of (PF) from theorem 3.9. Then there exists an increasing sequence $k(j) \to \infty$, $j \in \mathbb{Z}^+$, as above, and a solution (λ, φ) of (P) satisfying (1), (4), and (6) as well as

(2') $\|\varphi\|_{W^{1,2}(\Omega)} < C\{\lim_{j \to \infty} \|\varphi_{k(j)}\|_{k(j)}\} < \infty,$

(3') $\|\varphi\|_{C^{2,\omega}} < C(\delta + \delta^n)^\omega$, where $\delta = \dfrac{(\mu-\lambda)^{\frac{5-m}{4(m-1)}}}{(\lambda\alpha-\sigma)^{\frac{1}{(m-1)}}}$.

<u>Proof</u>: The bounds derived up to this point enable one to extract a subsequence of solutions converging to the desired solitary wave solution. As in lemma 4.7 of Bona, Bose, Turner (1983), it can be shown that $\|\varphi_k\|_{L^\infty}$ is bounded below by a positive constant independent of k. It follows that $\varphi > 0$. For further details see theorem 5.1 in Bona, Bose, Turner (1983).

6. <u>Example</u>

In the following specific example we show how the theory developed in the previous sections is applied.

For the primary flow, $\vec{q} = (0,V(r),W(r))$, we take $W(r) = d$, $V(r) = \Omega r^2$, where d is a constant and Ω is an angular velocity parameter. As noted by Pritchard (1970) and Benjamin (1962), solitary waves do not exist when the flow is a rigid body motion, $V = \Omega r$. In the following calculation $V = \Omega r$ would produce a linear equation, while we require a nonlinear equation. Since we are looking for permanent waves, say of velocity $\bar{c}$, travelling down the cylinder, we change to a coordinate system moving with velocity $\bar{c}$. This renders the wave form stationary; $W(r) = d$ is replaced by $\bar{W}(r) = d - \bar{c} = c$. Using (2.2),

$$\Psi(r) = \int_0^r s\bar{W}(s)ds = \frac{cr^2}{2}$$

is the stream function for the unperturbed flow. Inverting this we have $R^2 = \frac{2\Psi}{c}$. Then $K(r) = rV(r) = \Omega r^3$ and in the primary (unperturbed) flow $I(r) = \frac{K^2(r)}{2}$ becomes $I(\Psi) = \frac{4\Omega^2}{c^3}\Psi^3$. Hence $\frac{dI}{d\Psi} = \frac{12\Omega^2\Psi^2}{c^3}$. Defining $\varphi(r,z)$ by (2.8) where

$$\psi(r,z) = \Psi(r) + \gamma\varphi(r,z), \tag{6.1}$$

ψ is the stream function for the perturbed flow, φ is the perturbed stream function

and γ is a normalizing constant to be chosen shortly. We note that $\Psi_{rr} - \frac{\Psi_r}{r} \equiv 0$, so from (2.9) we have

$$\gamma(\varphi_{rr} - \frac{1}{r}\varphi_r + \varphi_{zz}) = \frac{dI}{d\Psi}(\Psi+\gamma\varphi)\left[\frac{R^2(\Psi)}{R^2(\Psi+\gamma\varphi)} - 1\right]$$

$$= \frac{12\Omega^2}{c^3}(\Psi+\gamma\varphi)^2\left[\frac{2\Psi}{c}\cdot\frac{c}{2(\Psi+\gamma\varphi)} - 1\right]$$

$$= \frac{12\Omega^2}{c^3}(\Psi+\gamma\varphi)[\Psi - (\Psi+\gamma\varphi)]$$

$$= -\frac{12\Omega^2}{c^3}\gamma\varphi(\Psi+\gamma\varphi)$$

$$= -\frac{6\Omega^2}{c^2}\gamma\varphi\left(r^2 + \frac{2\gamma}{c}\varphi\right).$$

Hence

$$-\left(\varphi_{rr} - \frac{1}{r}\varphi_r + \varphi_{zz}\right) = \frac{6\Omega^2}{c^2}\left[r^2\varphi + \frac{2\gamma}{c}\varphi^2\right]. \qquad (6.2)$$

We have an equation of the form

$$-\left(\varphi_{rr} - \frac{1}{r}\varphi_r + \varphi_{zz}\right) + h(r,\varphi) = \lambda f(r,\varphi)$$

where $h(r,\varphi) \equiv 0$, $\lambda = \frac{6\Omega^2}{c^2}$ and $f(r,\varphi) = r^2\varphi + \frac{2\gamma}{c}\varphi^2$. Note that λ is proportional to Ω^2 and inversely proportional to c^2, the square of the velocity of the wave of permanent form relative to the velocity of the primary flow. Choosing γ by $\gamma = c/2$, we have two cases. They correspond to $c > 0$ and $c < 0$.

Consider first $\gamma > 0$. We remember that in deriving (2.9), (2.10) we followed stream surfaces to $z = \pm\infty$ and used the knowledge that the Bernoulli quantity, H, and the "circulation", I, were constant on stream surfaces. So the f and h defined from (2.9), (2.10) require that ψ takes its values in $[0,\Psi(a)]$. In our example this is $[0,ca^2/2]$. Since $\varphi(0,z) = \varphi(a,z) = 0$, this requirement is implied by

$$\psi_r = \Psi_r + \gamma\varphi_r > 0 \qquad (6.3)$$

for $(r,z) \in (0,a) \times \mathbf{R}$. Using $\Psi_r = cr$ and $\gamma = c/2$, we see that (6.3) is equivalent to

$$\varphi_r > -2r. \qquad (6.4)$$

The restriction (6.4) will certainly be satisfied if we can verify that for our solution

φ,

$$|\varphi_r| < 2r, \tag{6.5}$$

for $(r,z) \in \Omega = [0,a] \times \mathbb{R}$. In the perturbed flow the radial velocity $u(r,z) = -\frac{\psi_z}{r} = -\frac{\gamma\varphi_z}{r}$. We know our solutions are positive $(\varphi = \hat{\varphi})$ and have a profile in z like e^{-z^2}. In the region to the left of the crest, $\varphi_z > 0$, so $u(r,z) < 0$ and we have a wave of depression. As stated above, $f(r,t) = r^2t + \frac{2\gamma}{c}t^2 = r^2t + t^2$ is, at the moment, only valid for $r \in [0,a]$ and $t > 0$ such that $\Psi(r) + \gamma t$ lies in $[0,\Psi(a)]$. So (r,t) must satisfy

$$0 < t < \frac{\Psi(a) - \Psi(r)}{\gamma} = \frac{1}{\gamma}\left(\frac{ca^2}{2} - \frac{cr^2}{2}\right) = a^2-r^2. \tag{6.6}$$

However, by extending $f_1 = t^2$ as an odd function to $t < 0$, we see that condition (H) of section 3 is clearly satisfied with $m = n = 2$, $\alpha = d = \frac{2\gamma}{c} = 1$, and $\theta = 2/3$. These conditions are satisfied without any restrictions on the values of $t > 0$, so our extension of $f = r^2t + t^2$ outside of $0 < t < a^2-r^2$ is permissible.

Since the f and h derived for our example satisfy (H), theorems 3.7 and 3.9 give the existence of z-periodic solutions (λ,φ) of equation (3.1). In (PC) $R > 0$ is specified and in (PF) $\lambda \in (0,\mu)$ is specified. We know that $\varphi = r^2g$, where $g \in C^2$ and $r < a$, so $\varphi_r = 2rg + r^2g_r$. Hence

$$|\varphi_r| < 2r|g| + r^2|g_r| < 2r\left(1 + \frac{a}{2}\right)\|g\|_{C^1}. \tag{6.7}$$

From corollary 4.5 in Bona, Bose, Turner (1983), with $N = R$ in (PC) and

$$N = \frac{(\mu-\lambda)^{\frac{5-m}{4(m-1)}}}{\lambda^{\frac{1}{m-1}}} \text{ in (PF)}, \quad \|g\|_{C^1} < C(N+N^2).$$

Hence (6.6) will be satisfied if $(1 + a/2)\|g\|_{C^1} < 1$. This can be accomplished if we restrict R to $R \in (0,R_0)$ for some $R_0 > 0$ or if we restrict λ to $\lambda \in (\lambda_0,\mu)$ for some $\lambda_0 \in (0,\mu)$. With these restrictions

$$\psi(r,z) = \Psi(r) + \gamma\varphi(r,z) = c(r^2 + \varphi)/2, \quad c = \left(\frac{6\Omega^2}{\lambda}\right)^{1/2},$$

has range in $[0,\Psi(a)]$, is a solution of (2.5) and gives, for each $k > 0$, a train of depression waves. Since $m = 2 < 5$ and $0 = \sigma/2\alpha < \mu$, lemma 4.1 of Bona, Bose, Turner

(1983) shows $\lambda < \mu$ when R is given. Inequality (5.1) guarantees that φ has nontrivial z-dependence for k sufficiently large. Finally, theorem 5.2 gives the existence of solitary wave solutions (λ, φ) with $\psi = c(r^2 + \varphi)/2$. Since $|\lambda f_1 - h_1| = \lambda t^2$ we know that the deviation of the velocity fields of these solitary waves from the velocity field of the primary flow decays exponentially as $|z| \to \infty$.

In the case $c > 0$, the fluid is moving downstream as viewed from the stationary waveform $(\overline{W} > 0)$. In considering the case $\gamma = c/2 < 0$, we note that the change in sign of c corresponds to the fluid now moving upstream relative to the waveform $(\overline{W} < 0)$. Hence, the case $c < 0$ is just the reflection, in $z = 0$, of the case $c > 0$. Therefore, we again obtain a wave of depression.

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Solitary and periodic internal waves are shown to exist in swirling flow. Incompressible, inviscid fluids in a right cylinder of infinite length and finite radius are considered. Variational techniques are used to demonstrate that the Euler equations possess solutions that represent progressing waves of permanent form. Moreover, internal solitary wave solutions are shown to arise as the limiting forms of internal periodic waves as the period length becomes unbounded.

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